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A Dynamic Analysis Method for Complex Cable Systems Based on Dynamic Stiffness Method

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Abstract: Cable structure is an important type of structural bearing system. With the increasing of span and height of modern engineering structure, the length of cable used in civil engineering is increasing, and its structural form becomes more and more complex. In this case, the dynamic analysis of cable structure has become the key to the structural design, service performance monitoring and maintenance, and vibration control. The existing cable dynamic analysis theory can only consider the influence of some design parameters, or be suitable for cable systems with relatively simple structural forms, and difficult to balance analysis efficiency and accuracy and apply to the exact dynamic analysis of complex cable structures. Based on the dynamic stiffness method (DSM), this paper develops a set of exact dynamic analysis method for complex cable system; The effectiveness and applicability of this method are illustrated by taking naked cable system, composite cable system and multi-segment cable system with lateral supports as representatives. The main contributions and achievements are as follows: The exact solution of the frequency equation of small sagged cable considering bending stiffness, internal damping and additional cable force is presented; The dynamic modeling and solution of stay cables with double-sheathing anticorrosion system are given, and the influence of filling material on damping and frequency of the system is explored; Based on the DSM, the dynamic model of multi-segment cable system is established, and the corresponding dynamic solution scheme is proposed. According to this, the dynamic characteristics of the main cable of a suspension bridge are analyzed. The theoretical analysis results are in good agreement with the measured values and finite element solutions, which shows the accuracy of the proposed method. The cable dynamic analysis method proposed in this paper has high calculation accuracy and is easy to implement, which can provide theoretical basis for the initial design and health monitoring of cable and cable structures in engineering.

Keywords: Cable structure; Structural dynamics; Dynamic stiffness method; Structural health monitoring; Exact dynamic analysis

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1. Introduction

Cable structure is an important type of steel structure load-bearing system, which is widely used in various types of steel structures such as large industrial factories, large venues, high-rise and large-span bridges due to its beautiful architectural shape and good structural performance. With the continuous growth of the span and height of modern engineering structures, the length of structural cables continues to grow, and their structural forms have become increasingly complex. Whether it is a single cable or a cable structure composed of the main load-bearing component - the cable, a common prominent problem is that as the length increases, the slenderness ratio of the cable becomes larger, and the transverse stiffness and internal damping are very small. Under the influence of wind load, moving load, etc., various types of vibration phenomena are easily generated [1, 2], resulting in alternating stress in the internal steel wire. This vibration phenomenon not only leads to dynamic stability and safety issues of the structure, but also affects the fatigue durability and applicability of the structure. In this case, the dynamic analysis of cable structures has become a key issue in the design, performance monitoring and maintenance during service, and vibration control of such structures [3-5].

As the mechanical behavior of cable structures is characterized by large deformations and small strains, the geometric nonlinearity is very significant, and the vibration control equation of the structure is nonlinear, which makes the dynamic analysis theory of cable structures very complex. Cable dynamics has a long research history and has formed multiple research branches, including wind induced vibration and aerodynamic stability of cables, parametric vibration of cable-beam systems, and vibration control of cables. However, these topics are all based on the research of cable dynamic characteristics. It can be seen that in order to achieve refined dynamic analysis of cables, it is necessary to review and summarize existing dynamic analysis theories, and make improvements to address their shortcomings. The existing dynamic analysis theories of cable structures can be divided into analytical methods, numerical methods, and semi-analytical and semi-numerical methods according to their different analytical methods and solution forms.

The analytical method treats the cable system as a continuous system with infinite degrees of freedom. By solving the control differential equation of the system, the vibration mode function of the cable is determined, and the characteristic frequency equation and motion equation of the system are obtained. By solving this equation, the analytical form of dynamic characteristics and response can be obtained. In practical engineering, the tension string model-based dynamic analysis method is essentially an analytical method [6-8]. Starting from the tension string model, an analytical form of cable force identification formula can be obtained. However, due to the fact that the actual cable itself has a certain bending stiffness, the string theory-based dynamic analysis method cannot always take into account the influence of cable bending stiffness, resulting in the conclusions obtained will inevitably deviate from the actual situation. Therefore, many scholars have used the simple supported beam model to discuss the relationship between cable force and natural frequency under the influence of factors such as cable sag, bending stiffness, and boundary conditions [9-12]. Numerical calculation methods can be generally divided into two categories: the first category is discrete numerical calculation methods, that is, the discrete system is used to approximate the original continuous system to obtain approximate solutions, such as the finite element method [13], the weighted residual method [14], and the finite difference method [15]; The other type is continuous numerical calculation methods based on continuum mechanics and wave theory. The characteristic of this method is to directly solve the equilibrium differential equation or wave equation of the continuous quality system, and use the complex exponential function as the interpolation function, so as to obtain the accurate solution of the continuous quality system, such as the spectral finite element method [16, 17]. Whether it is the finite element method, finite difference method, or the meshless method that only needs node information, it is inevitable that the cable structure needs to be discretized, and this process will inevitably introduce errors; In order to reduce the error and improve the analysis accuracy, it is necessary to increase the degree of discretization at the expense of the calculation efficiency, which means that the structure discretization processing needs to be repeated after each parameter adjustment. The process will seriously affect the efficiency, and it is difficult to give consideration to both the calculation accuracy and the calculation efficiency, so it is not a good solution.

The semi-analytical and semi-numerical method combines the advantages of both analytical and numerical methods. In structural modeling, there is no need to discretize the structure or approximate the vibration mode function of the structure, thus faithfully reflecting the dynamic behavior of the structure. By numerically solving the characteristic frequency equation and dynamic equation of the structure, accurate numerical solutions of the structural dynamic characteristics and response can be obtained. It follows that the search for an accurate analysis theory with semi-analytical and semi-numerical values is essential for precise dynamic analysis of actual engineering structures. The Dynamic Stiffness Method (DSM) is a semi-analytical semi-numerical method that starts from the precise vibration mode function of the system to obtain the frequency equation and the dynamic equation in closed form. When the form of both is relatively simple, an analytical solution can be obtained. In most cases, numerical calculation methods are needed to obtain numerical solutions. The authors of this paper have given closed-form solutions for the transverse dynamic stiffness of Euler beams with small droop based on dynamic stiffness method [18, 19], and studied the variation of transverse dynamic stiffness in space and frequency domain; Subsequently, [20] further provided a unified characteristic frequency equation and its numerical solution for the structure under the combined influence of lateral force elements and sag effects. At the same time, a multi-level and multi-parameter identification algorithm suitable for cable-stayed cables was proposed based on the PSO optimization algorithm [21]; Due to the fact that the cable is usually wrapped with a sheath and a grease filled layer, it is essentially a composite structure rather than a homogeneous structure. For this reason, Han Fei and Dan Danhui et al. [22] used the Euler beam theory to model the composite cable using a double beam system model connected by distributed springs, derived the dynamic stiffness matrix of the model, and studied the influence of parameters such as sag, spring connection layer stiffness, and double beam bending stiffness ratio on the system modal frequency. All these research works have laid the foundation for the application of dynamic stiffness method in cable dynamics.

This article will focus on three typical types of cable support structures in practical engineering: (1) Bare cable systems that simultaneously consider factors such as cable bending stiffness, sag, boundary conditions, and internal damping; (2) Steel cables with double layer anti-corrosion sheath; (3) A multi-segment cable system with multiple lateral supports. In order to achieve the refined dynamic analysis of the typical cable structure mentioned above, this article will use dynamic stiffness method. However, there are some problems in the original DSM that has not been able to solve when applied to cable dynamic analysis: (1) the original DSM can only be used for undamped systems, and the original method will no longer be effective when considering the damping characteristics of the cable or filling layer; (2) The original DSM method can only be used for linear system, and when considering the influence of cable sag, its dynamic stiffness matrix will no longer be symmetrical, so it is difficult to accurately solve the frequency equation of the system. In view of this, this article will first give a brief review of DSM and point out the technical difficulties that need to be addressed when analyzing complex cable structures. Subsequently, taking three typical cable systems as examples, we will introduce how to apply DSM for dynamic analysis. Finally, this paper concludes and summarizes the general dynamic analysis approach for cable supported bridge structures. Based on the dynamic stiffness method and the extended Wittrick-Williams algorithm, this article calculates the dynamic characteristics of various types of cables and forms a set of high-precision and widely applicable dynamic analysis theories for complex cable structures, which can provide theoretical basis for the design, health monitoring during operation, and vibration control of complex cable structures.

2. Introduction to Basic Methods

2.1. Introduction to the Dynamic Stiffness Method

The dynamic stiffness method was proposed by Koloušek in the 1940s as an efficient, accurate, and stable structural dynamic analysis method [23]. Dynamic stiffness is another expression describing the control differential equation (GDE) of a structure, whose solution satisfies the control differential equation at any point on the structure and all equilibrium conditions, displacement coordination, or constraint conditions at the nodes. Generally, DSM is established based on the strong form solution of the structural control differential equation (GDE) under boundary condition (BC) constraints. It starts from the control differential equation in the frequency domain of the system and obtains the accurate form function of the system by finding its general solution, thereby obtaining the equilibrium equation in the form of dynamic stiffness of the system. The established dynamic stiffness matrix is a closed function of frequency, which can be used to calculate modes of any order without depending on the discretization of the structure, so its calculation efficiency and accuracy are better than those of the finite element method. This property of DSM enables it to be applicable to any boundary condition and frequency range, especially in high-precision and high-order modal solutions; It can accurately obtain the dynamic characteristics of the structure over a wider frequency range, without sacrificing computational efficiency; At the same time, DSM method can also be extended to complex engineering structures by means of finite element mesh generation and singleton group method.

There are two key issues in the implementation of DSM: (1) the establishment of accurate dynamic stiffness matrix and frequency equation; (2) The accurate solution of frequency equation. For the first problem, in order to obtain the dynamic stiffness matrix of the structure, it is necessary to first establish the motion differential equation of the structure, and this part of the work can be achieved by referring to corresponding literature or classical theory. Generally, the derivation process of the dynamic stiffness matrix of undamped structures can be referred to in the literature [24], and this article will not elaborate further. The second key issue usually directly determines the applicability of DSM. Although DSM is an accurate solution, its efficiency and accuracy are achieved by increasing the approximation degree of the shape function to the actual structure and the difficulty of solving the frequency equation. The accurate shape function of the structure can be obtained through two efforts. One is to ensure that the general solution of the control differential equation (system) can be analytically given, and the other is to ensure that the solution of the frequency equation of the structure does not miss roots or add false roots. The mathematical difficulties brought by the above two conditions also to some extent limit the application scope of DSM. In order to achieve accurate solution of the frequency equation, it is necessary to introduce the Wittrick-Williams (W-W) algorithm.

2.2. Introduction to the Dynamic Stiffness Method

The calculation of structural modal frequency is one of the cores of structural dynamic analysis, which can be solved by solving the structural characteristic frequency equation established by the DSM. In earlier studies, some mathematical techniques were used to solve frequency equations for specific simple structures, such as the work of Timoshenko [25, 26] and Huang [27]. In order to solve the frequency equation of complex systems, scholars such as Cheng [28], Li [29], and Henshell [30] proposed the "graphical method", which essentially determines the modal frequencies at each order of the structure through zero-root search method. This type of method not only requires a large amount of computation, but also has drawbacks that it is difficult to distinguish the two modal frequencies that are very close to each other. Due to the difficulty in distinguishing the sign changes at zero and positive and negative infinity, it may lead to missing or increasing roots [31]. As a result, the mode can be lost or mixed with false modes. The Muller search method [32] belongs to this category of methods.

For this reason, Wittrick and Williams proposed the Wittrick-Williams (W-W) algorithm in the 1970s [31, 33-35] for analyzing structural free vibration and buckling problems [36] and wave conduction problems [37]. Due to the wide applicability of the W-W algorithm, researching on its application and improvement has become a hot topic in DSM since then. Zhong used Rayleigh's theorem and the properties of Sturm sequences to generalize the W-W algorithm [38], theoretically solving the problem of missing roots in solving frequency equations, and solving the bottleneck problem that has plagued the DSM method for many years, promoting the development and engineering application of DSM theory. In fact, the W-W algorithm is not directly used for frequency calculation, but rather obtains the number of modal frequencies of the structure that are less than the frequency of the test by counting, thereby determining the upper and lower bounds of any order frequency. Finally, combined with the dichotomy or Newton method, the frequency solution with any accuracy is obtained. Due to its ability to maintain the Sturm sequence characteristics of the dynamic stiffness matrix and ensure that roots are not omitted during the equation solving process, its stability and correctness have also been theoretically proven, which is an advantage that most analytical and approximate methods do not possess [34].

Although the W-W algorithm can solve the problem of accurately solving the frequency equation of DSM, it also has certain limitations. For example, when the structural form is complex or considering the effects of internal damping and geometric nonlinearity, the W-W algorithm will be difficult to directly use. Therefore, the following chapters of this article will provide improvement strategies for DSM and W-W algorithms for three typical types of cable structures, in order to form a universally applicable theory for cable dynamics analysis.

3. Engineering Applications

3.1. A Bare Cable System Considering Multiple Factors

Bare cable system refers to a single cable structure with no other transverse force or component action along the cable length direction except for end restraint. Common projects include cable without damper, suspender without transverse support and power transmission cable. The structural form of this type of cable is relatively simple, and existing research has been able to consider the influence of some factors such as structural bending stiffness, sag, inclination angle, and internal damping, but it is not yet possible to comprehensively consider these factors in an analytical or semi-analytical form. In view of this, this article establishes a bare cable model based on the Euler beam theory as shown in Figure 1:

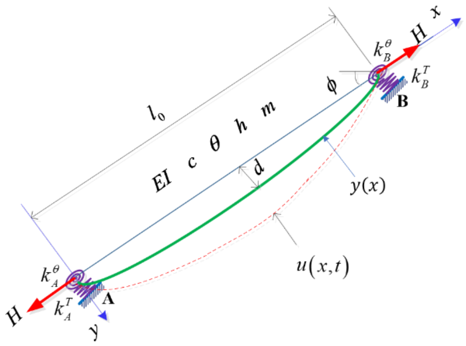


Figure 1. Naked cable model considering multiple parameters.

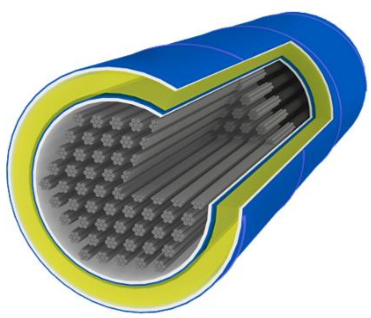


Figure 2. Cable with double sheathing

Figure 2 is a bare cable mechanical model that considers the influence of factors such as cable line mass m , bending stiffness EI, sag d , inclination angle ϕ , additional cable force $h(t)$, and damping coefficient c . Where l_0 the chord length of the cable, H is the cable force, $y(x)$ is the initial static configuration of the cable, $u(x,t)$ is the lateral displacement function of the cable, that is, the dynamic configuration, $k_{A(B)}^{\theta}$ and $k_{A(B)}^{T}$ is the rotational stiffness and vertical support stiffness when the boundary is elastically supported. At this point, the control differential equation of the system can be expressed as:

$$EI\frac{\partial^4 u}{\partial x_j^4} - H\frac{\partial^2 u}{\partial x_j^2} - h(t)\frac{d^2 y}{dt^2} + m\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} = 0 \tag{1}$$

In the above equation, x_j is the segmented coordinate, $y(x)$ is the configuration of the cable at the static equilibrium position. For cables considering the influence of bending stiffness, their static configuration generally follows the quadratic parabolic assumption, i.e.$y(x) = \frac{-4e}{l_0}x(x-l_0)$, $h(t) = \frac{EA}{l^e}\int_0^{l_0}\frac{\partial u}{\partial x}\frac{dy}{dx}dx$ is the additional dynamic cable forces caused by elastic elongation of the cable, and the sag $e = \frac{mgl_0\cos\theta}{8H}$ and $l^e = \int_0^{l_0}\left(\frac{ds}{dx}\right)^3 dx \cong l_0(1+8e^2)$ is the effective length of the cable [39].

It can be seen that equation (1) has two additional terms, damping force $c\frac{\partial u}{\partial t}$ and nonlinear dynamic cable force $h(t)\frac{d^2 y}{dt^2}$, compared to the free vibration equation of undamped Euler beams, the presence of which will directly lead to the inability of the original DSM to be used directly. Therefore, the author of this article proposes an extended dynamic stiffness method (EDSM) [40] and an improved W-W algorithm to consider the effects of damping and nonlinear dynamic cable forces, respectively. The advantage of EDSM is that we can still calculate the dynamic stiffness matrix and modal frequency ω_0 of the corresponding undamped system using the DSM method, and then use EDSM to derive an analytical expression for the modal damping ratio of the damping system. Applying EDSM to equation (1) can obtain the damping frequency $\omega_D = \sqrt{1-\xi^2}\omega_0$ of the system considering the internal damping of the cable, where the damping ratio

$$\xi = \frac{c}{2m\omega_0} \tag{2}$$

However, during the process of solving ω_0, due to the influence of nonlinear terms $h(t)$, the dynamic stiffness matrix of the system is no longer strictly symmetrical, making it impossible to directly apply the W-W algorithm for frequency calculation. In view of this, the author provides a frequency correction formula for considering additional cable forces and small sag based on perturbation methods: variables are separated by substituting $u(x,t) = \sum_{n=1}^{\infty}\phi_n(x)q_n(t)$ into equation (1), and based on the principle that the work done by the dynamic cable force $\eta q_n\int_0^l \phi_n dx$ and its equivalent linearized force $\eta q_n\phi_n\gamma_n$ in one cycle is equal, we can obtain the proportional coefficient $\gamma_n = \left(\int_0^l \phi_n(x)dx\right)^2 / \int_0^l \phi_n^2(x)dx$, where $\eta = \frac{EA}{l^e}\frac{dy}{dx}$. From this proportional coefficient, we can further derive the correction formula for the modal frequency of small sag cables considering nonlinear effects:

$$\omega_0 = \sqrt{\omega_m^2 + \eta\gamma_n} \tag{3}$$

Where ω_m is the modal frequency of the undamped linear system ($c = h = 0$).

By using equations (2) and (3), an accurate solution of the modal frequency for a bare cable system considering multiple factors can be achieved. The undamped linear frequency ω_m can be obtained from DSM based on the Euler beam model, while the damping ratio ξ and the corrected frequency ω_0 considering the influence of dynamic cable force can be given by EDSM and perturbation method, respectively. At this point, this article utilizes DSM and the proposed EDSM to accurately solve the dynamic characteristics of the small sag bare cable system, and subsequent dynamic response analysis can also be further achieved on this basis.

3.2. Composite Cable System

In order to protect the steel cable from rust, it is necessary to carry out the necessary protection for the made steel cable, and the parallel steel strand clue can be roughly divided into two common forms of protection from the anti-corrosion system: (1) single-layer cable sleeve anti-corrosion system. This process is based on PE (polyethylene) as the main body, adding 2.6% carbon black and other additives, and the PE plastic is directly squeezed on the cable under heating conditions, which is a common anti-corrosion measure in civil engineering; (2) double-layer cable sleeve anti-corrosion system. As shown in Figure 3, the strands coated with anti-rust grease are first bundled and passed through high-density polyethylene (HDPE) casing to form a steel bundle, and then the entire steel bundle is threaded into a larger HDPE casing, and the middle is filled with flexible filler such as grease. For the single-layer cable system, because the casing and the steel bundle are closely connected, the two can be regarded as a whole during the vibration process, and many studies have used the tensioned chord and single beam models to analyze the dynamic characteristics of the system, but the more complex double-layer cable sleeve anti-corrosion system has made the tensioned string theory unable to correctly reflect the dynamic characteristics of the cable.

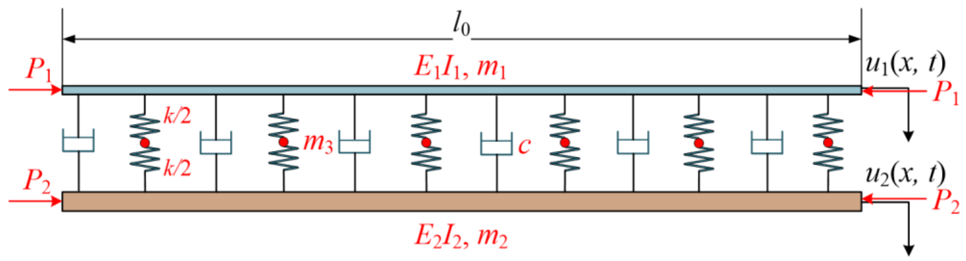


Figure 3. Schematic diagram of the double beam model

The relevant literature has pointed out that the HDPE casing of the cable in the operation stage has a non-negligible impact on the cable force test [41], in the actual frequency measurement process, the outer casing will have a greater impact on the measurement quality and accuracy of the cable-stayed frequency, which is manifested as: it is not possible to accurately determine whether the frequency measured by the sensor installed on the casing is dominated by the contribution of the steel bundle or the contribution of the sheath, which will directly affect the accuracy and correctness of the frequency method of the traditional single girder model. Therefore, it is necessary to start from the analysis of the dynamic characteristics of the double beam system and study a new frequency method to measure the force of the cable. There is still a vacancy in this area of work due to the following reasons: (1) The difficulty of establishing theoretical models. Compared with the single-beam system, how to model the double-beam system is a difficult point. (2) The complexity of theoretical solutions. Since the material and structural parameters of the casing and filling layer directly affect the dynamic characteristics of the system, the establishment and solution of the dynamic balance equation of the system is bound to be more complicated. In view of this, the authors of this paper establish the double-beam model shown in Figure 3 to model the dynamics of double-layer cable lasso, the length of No. 1 beam (Beam1) and No. 2 beam (Beam2) is l , which are used to simulate the outer HDPE sheath and the inner cable respectively, the cross-sectional flexural stiffness and unit length line mass of the two are E_1I_1, m_1, E_2I_2, m_2, and the effective cross-sectional areas are A_1 and A_2 respectively. The filling layer directly from the double-layer cable sleeve is simulated with a viscoelastic connection layer with a stiffness of $k(Nm^{-2})$, a damping coefficient of c (Nsm^{-1}), and a linear quality of m_3. $u_1(x,t)$,$u_2(x,t)$ respectively represents the displacement function of beam elements 1 and 2 under the coordinate system x, and according to Hamilton's theorem, the control differential equation of the double beam system under the action of axial force can be expressed as follows

$$\begin{cases} E_1I_1\dfrac{\partial^4u_1}{\partial x^4}+\dfrac{m_3}{4}\left(\dfrac{\partial^2u_1}{\partial t^2}+\dfrac{\partial^2u_2}{\partial t^2}\right)+m_1\dfrac{\partial^2u_1}{\partial t^2}+k(u_1-u_2)+c\left(\dfrac{\partial u_1}{\partial t}-\dfrac{\partial u_2}{\partial t}\right)+P_1\dfrac{\partial^2u_1}{\partial x^2}=0 \\ E_2I_2\dfrac{\partial^4u_2}{\partial x^4}+\dfrac{m_3}{4}\left(\dfrac{\partial^2u_1}{\partial t^2}+\dfrac{\partial^2u_2}{\partial t^2}\right)+m_2\dfrac{\partial^2u_2}{\partial t^2}-k(u_1-u_2)-c\left(\dfrac{\partial u_1}{\partial t}-\dfrac{\partial u_2}{\partial t}\right)+P_2\dfrac{\partial^2u_2}{\partial x^2}=0 \end{cases} \tag{4}$$

Equation (4) considers the influence of damping and quality of the flexible filler layer, and we can still use EDSM to achieve its accurate solution, that is, substituting$u_i=\varphi_i(x)e^{\alpha t}, i=1,2$ to Equation (4) and separating the variables, so that the damping system and its corresponding undamped system have a unified form of dynamic stiffness matrix, and the system damping ratio can be obtained by applying EDSM again [42]:

$$\xi=\frac{c(B-A)}{(m_2B-m_1A)\omega_0} \tag{5}$$

It can be seen from the Equation (5) that the damping characteristics of the double-layer cable sleeve composite cable system can be improved by optimizing the material characteristics of the filling layer. Since the stiffness coefficient k of the viscoelastic layer has negligible influence on the damping ratio of the system compared with the mass m_3 and damping coefficient c, it is unreasonable to change the damping ratio of the structure by increasing k. In view of this, this section discusses the influence of dimensionless mass parameters $\bar{m}_3=m_3/m_1$and dimensionless damping coefficient $\bar{c}=\frac{cl^2}{\sqrt{m_1E_1I_1l}}$of viscoelastic layers on the damping characteristics of composite cables to illustrate the application of the Equations (4) and (5) in practical engineering.

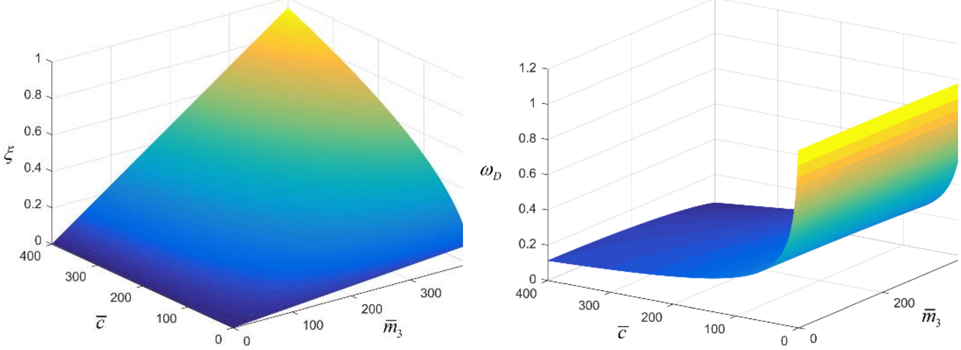
In the actual application process, the outer cable sleeve and inner steel bundle of the cable are simulated by No. 1 beam and No. 2 beam respectively, because the sheath does not bear the axial force, make $P_1=0$; The filler layer is simulated as a viscoelastic layer, which has much less mass and stiffness than the steel bundle. The design parameters of the structural materials are shown in Table 1.

Table 1. Design parameters of double- sheathing cable system

$\boldsymbol{m_1}$	$\boldsymbol{m_2}$	$\boldsymbol{P_2}$	$\boldsymbol{E_1I_1}$	$\boldsymbol{E_2I_2}$	$\boldsymbol{l}$	$\boldsymbol{m_3}$	$\boldsymbol{k}$
0.2kgm^{-1}	0.76kgm^{-1}	1×10^6 N	0.2 Nm^2	16.6 Nm^2	100m	0.01kgm^{-1}	10Nm^{-2}

Figure 4 shows the variation pattern of the first-order modal damping ratio and frequency of the system when $\bar{m}_3$ and $\bar{c}$ change from 1 to 400. It can be seen that compared to the linear mass $\bar{m}_3$, the damping coefficient $\bar{c}$ of the viscoelastic layer has a more significant impact on the modal frequency and damping ratio, and optimizing the damping coefficient can effectively improve the damping characteristics of the system.

This section establishes a dynamic model for a type of double- sheathing cable system in engineering, taking into account the damping, mass and stiffness of the filling layer, as well as the influence of axial force and boundary factors. The precise solution of the system's dynamic characteristics is achieved using EDSM and W-W algorithms. The influence of material characteristics of the filling layer on the damping ratio and frequency of the system was explored through practical cases, and the applicability of the research results in this section in practical engineering is also explained.



(a) (b)

Figure 4. Influence of material parameters of connection layer on first-order modal damping ratio of system: (a) The influence of $\bar{m}_3$ and $\bar{c}$ on the first-order modal damping ratio of the system; (b) The influence of $\bar{m}_3$ and $\bar{c}$ on the first-order modal frequency of the system

3.3 Multi-segment Lateral Support Cable

The cable system with multiple lateral supports is a common type of cable load-bearing structure in engineering, such as cable-auxiliary cable system, suspension cable-hanger system, and cable-damper system, etc. Influenced by the lateral supports, each cable section divided by it will follow a different dynamic configuration in the vibration process, the accurate calculation of the dynamic configuration of each cable section and the additional cable force is a necessary prerequisite for the fine dynamic analysis of the multi-segment cable system. As the number of transverse elements increases, the number of structural dynamic degrees of freedom will also increase rapidly, what makes the analysis and calculation of multi-segment cable systems more difficult. There is not yet a general analytical or semi-analytical dynamic analysis theory that can effectively consider the effects of cable flexural stiffness, sag, additional cable forces, and multiple transverse elements. In this section, the analytical ideas and research framework of multi-segment elastic support cables based on dynamic stiffness method will be given in order to provide a new research method for the refined dynamic analysis of this type of structures.

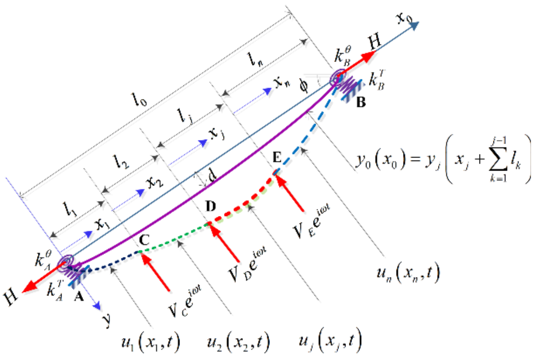


Figure 5. Multi-segment cable system divided by several lateral forces

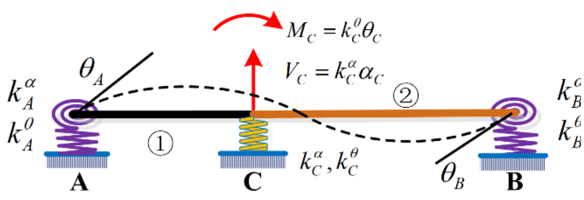


Figure 6. Mechanic model of multi-segment cable with intermedium forces and supports

As shown in Figure 5, considering the general boundary conditions, A and B at the end of the cable are elastically supported by the vertical springs and rotating springs, $k^{\theta}_{A(B)}$ and $k^{T}_{A(B)}$ represent the stiffness of the rotating springs and vertical springs respectively, and the rest of the symbols have the same meaning as above. Consider that the cable in Figure 5 is divided into n natural segments by the $n-1$ transverse elements, and the chordal length of the j^{th} segment is l_j ($j=0$ represents the overall coordinates, same as below). Let the transverse forces generated by the transverse elements installed at the ends of the j^{th} cable section during vibration be $V_D e^{i\omega t}$ and $V_E e^{i\omega t}$, where $i = \sqrt{-1}$ is the imaginary unit, ω is the natural frequency of the system, and t is the time. For the convenience of the analysis, the global coordinates of the system are given by (x_0, y_0), and the local coordinate system of each cable section is given by $\left(x_j, y_j\right)$.

When the cable performs free vibration, the control differential equation for each cable segment divided by the transverse force can be expressed uniformly as

$$EI\frac{\partial^4 u_j}{\partial x_j^4} - H\frac{\partial^2 u_j}{\partial x_j^2} - h_j\frac{d^2 y_j}{d{x_j}^2} + m\frac{\partial^2 u_j}{\partial t^2} = 0 \tag{6}$$

Where h_j is the value of the additional cable force due to the elastic elongation of the j^{th} cable section during the vibration. The detailed derivation process of h_j can be found in [43]. Applying the DSM to the system described by Equation (6), the unit dynamic stiffnesses $\boldsymbol{K}^{(j)}$ of the multi-segment elastic support cables can be obtained, and in the modeling process, the action of the lateral elements or supports on the cables can be equal to the lateral concentrated force acting at the natural nodes, which will be related to the displacement amplitude of the support point. Once the dynamic stiffness matrix of each cable section is obtained, the overall stiffness matrix can be integrated according to the same law as the finite element, and the corresponding system frequency equation is derived. Then, the modal frequencies and mode shapes of the system can be obtained with the help of W-W algorithm. In practical engineering, many structures can be equated or simplified to a multi-segment elastically supported cable structure, such as the main cable of a suspension bridge, which is illustrated below with a single-span ground-anchored suspension bridge as an example.

In the analysis of free vibration of suspension bridges, the structural displacements are usually divided into in-plane vibration, out-of-plane vibration and torsional vibration. The displacement of in-plane vibration of suspension bridges is usually not coupled with other vibration, so this paper takes in-plane vibration of suspension bridges as an example and makes the following assumptions in the analysis:

(1). The main cable is of equal cross-section, and in a quadratic parabolic configuration under static load. The gravity of the main cable is assumed to be uniformly distributed along the span direction.

(2). Ignore the axial deformation of the main tower, the vertical stiffness at the top node of the tower is infinite, and there is no relative displacement between the main cable and the bridge tower.

Taking a suspension bridge shown in Figure 7 as an example, the rise span ratio of the bridge is 1:9.96, the side span length is 470 m, and the rise height is 4.4175 m. The total length of the main beam is 1485.16 m, with a total of 91 pairs of main span suspension rods. The distance between the suspension rods is 16.1 m, and the tower height is about 210 m. The elevation layout is as follows:

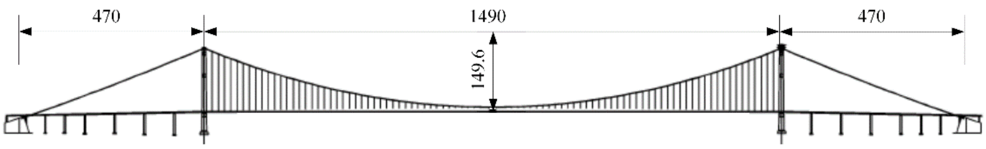


Figure 7. Elevation layout of a suspension bridge (Unit: m)

In order to compare with the simple methods, this paper conducts finite element modeling for the above suspension bridge, and on this basis, carries out numerical simulation research. Among them, when the general finite element analysis software ANSYS is used to establish the two-dimensional finite element model of the suspension bridge, the main beam is simulated by beam element Beam3, whose stiffness is the stiffness of the stiffening beam, and the density is the ratio of the sum of the dead loads of the first and second phases of the bridge deck system to the cross-sectional area of the stiffening beam, as shown in Table 2. The static configuration of the main cable and side cable is assumed to be a quadratic parabola. During modeling, the main cable and side cable are simulated using a planar beam element Beam3. Nodes are established at the suspension rod position, and the initial stress of the main cable is considered in the form of initial strain of the element in the completed bridge state. The design parameters are shown in Table 3. Unlike traditional finite element methods that use non compressed rod elements to simulate suspension rods, this paper will use a two-dimensional rod element Link 1 that can withstand uniaxial tension and compression to simulate suspension rods. The specific design parameters are shown in Table 4.

Table 2. Material and section parameters of the main girder

Item	Unit	Number
Elastic modulus	Pa	2.1×10^{11}
Sectional area	m^2	1.2935
Phase I Dead Load Conversion Density	kg/m^3	11.3112×10^3
Phase II dead load line density	kg/m	5.103×10^3
Conversion density of main beam	kg/m^3	15.2563×10^3
Conversion line mass of main beam (single cable bearing)	kg/m	9.867×10^3
Bending moment of inertia	m^4	2.0797

Table 3. Material and section parameters of the main cable

Item	Unit	Number
Angle of incidence of main cable	°	18.97
Side cable length (south/north)	m	496.61
Elastic modulus	Pa	2.0×10^{11}
Sectional area	m^2	0.5155
Line quality	kg/m	4052.4
Bending moment of inertia	m^4	0.0211
Horizontal component of main cable force	N	2.737433×10^8
Internal force of main cable anchor span	N	2.894656×10^8

Table 4. Material and section parameters of hangers

Item	Unit	Number
Elastic modulus	Pa	2.0×10^{11}
Sectional area	m^2	4.28×10^{-3}
Density	kg/m^3	7.21×10^3
Number	pair	91
Minimum boom length	m	3.184

The main tower of suspension bridge is simulated by Beam3 unit, and the elastic modulus E of concrete tower is 3.5×10^{10} Pa, cross-sectional area 35.5129 m^2, tower height 210 m, unit length line mass taken as 9.075×10^4 kg/m。Usually, the main tower consists of two tower columns and multiple beams. Strictly speaking, the main tower is a variable cross-section beam. However, due to the purpose of this article to validate the analysis method, for the convenience of modeling the bridge tower as a constant cross-section beam, its moment of inertia is taken as 165.2917 m^4.

Boundary conditions have a significant impact on the dynamic characteristics and response of structures. In order to truly reflect the actual dynamic characteristics of the structure, it is necessary to accurately simulate the connections between components and the boundary conditions of the system according to the design drawings. Based on this, when establishing the finite element model of the entire bridge, the main cable and tower top degrees of freedom are all coupled, and the side cable is consolidated at the anchor position.

When analyzing the dynamic characteristics of the main cable, it is necessary to simplify the suspension rod and main beam appropriately in order to avoid solving the complex cable-beam-suspension rod coupling vibration system. This article determines the vertical equivalent support stiffness provided by the main beam by calculating the flexibility of the main beam at each suspension rod position. Then, the suspension rod stiffness and the equivalent support stiffness of the main beam at that point are connected in series to calculate the vertical support stiffness of the main cable, and the main cable is equivalent to the multi segment elastic support cable structure studied in this section.

Table 5 presents the measured values, finite element solutions, analytical methods and calculation results using the method presented in this paper of the first twenty modal frequencies of the main cable. From Table 5, it can be seen that:

(1). Compared with the analytical method, the calculation results of this method are closer to the finite element solution. Except for the error of the 9th modal frequency of 7% (mainly because the modal frequency of this order is closer to the main beam, indicating a greater contribution of the main beam), the calculation error of other modes does not exceed 5%;

(2). The method proposed in this article has high computational accuracy in both low order and high order modes. Taking the fourth order mode as an example, the analytical method has a computational error of 6.14%, while the method proposed in this article only has 0.75%. In addition, for higher order modes, such as the 15th to 20th order modes, the calculation error in this article is within 1% (except for the 17th order mode dominated by the main beam);

(3). From Table 5, it can be further seen that the analytical method may lose some modal information in higher order, such as the 9th, 13th, and 17th modes. However, this method can effectively solve this problem. This indicates that the proposed method has better applicability than traditional analytical methods.

In general, the calculation results of this method are very close to the finite element and measured values, with high calculation accuracy and easy implementation, which can provide a theoretical basis for the initial design of suspension bridge main cables and health monitoring during the operation period, and is a reliable and worthy of promotion analysis method.

Table 5. Comparison of first 20 modal frequencies calculated by different methods.

Modal	FEM	Analytic method	Relative error (%)	Method of this paper	Relative error (%)	Measured value	Relative error (%)
1	0.0311	/	/	/	/	0.0278	/
2	0.0494	0.0499	1.01	0.0498	1.00	0.0500	0.21
3	0.0809	/	/	/	/	0.0916	/
4	0.1074	0.1009	6.14	0.1066	0.75	0.1139	6.42
5	0.1403	/	/	/	/	0.1417	/
6	0.1554	0.1540	0.97	0.1531	1.48	0.1667	8.15
7	0.2041	0.2101	2.89	0.1998	2.08	0.2148	6.96
8	0.2553	0.2701	5.72	0.2467	3.35	0.2389	3.28
9	0.3167	/	/	0.2938	7.24	0.3056	3.87

Modal	FEM	Analytic method	Relative error (%)	Method of this paper	Relative error (%)	Meas-ured value	Relative error (%)
10	0.3491	0.3347	4.12	0.3409	2.35	0.3417	0.24
11	0.3883	0.4044	4.14	0.3881	0.05	0.3694	5.06
12	0.4556	0.4799	5.33	0.4354	4.43	0.4528	3.83
13	0.5381	/	/	0.5304	1.43	0.5139	3.21
14	0.5561	0.5614	0.95	0.5781	3.95	0.5750	0.53
15	0.6272	0.6494	3.54	0.6259	0.21	0.5917	5.77
16	0.7182	0.7441	3.60	0.7219	0.51	0.7028	2.71
17	0.8026	/	/	0.7701	4.04	0.7722	0.26
18	0.8185	0.8457	3.33	0.8185	0.004	0.8417	2.75
19	0.9248	0.9545	3.21	0.9158	0.97	0.9444	3.02
20	1.0229	1.0705	3.13	1.0139	0.88	0.9833	3.11

4. Conclusions

Due to the large slenderness ratio and low lateral stiffness and internal damping of cables, their dynamic issues have become key to structural design, performance monitoring and maintenance during operation, and vibration control. In the existing theory of cable dynamic analysis, the dynamic stiffness method has been widely applied in cable dynamic analysis in recent years due to its accuracy and efficiency, and has achieved certain research results. However, due to the difficulty in solving the frequency equation, this method can only be used for simplified cable models that consider a few design factors. It is not suitable for bare cables comprehensively considered all design factors, composite cables composed of different materials, and cable-transverse component systems supported by transverse components in the middle.

In order to overcome the above difficulties and solve the technical difficulties faced in the dynamic analysis of complex cable systems, this paper establishes a set of high-precision and widely applicable dynamic analysis theories for complex cable structures. This theory effectively solves the problem of analyzing the dynamic characteristics of systems such as all-factor bare cables, multi-material composite cables, and multi-segment cables. The applicability and accuracy of the proposed analysis model and method were demonstrated through practical engineering cases. The results of this article can be further used for dynamic response analysis and parameter identification of cable structures, providing a powerful analytical tool for cable structure design, health monitoring during operation, and vibration control.

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